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CONCERNING THE EXISTENCE OF WALL TERMS IN A THREE-PARAMETER MODEL OF TURBULENCE

A. N. Volobuyev and A. P. Tolstonogov

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We show the existence of a wall term in the equation for $\langle u_1 u_2 \rangle$ in a three-parameter model of turbulence by analogy with the known terms in an "E-ε" model. We suggest a convenient system of differential equations for the boundary layer on a flat plate in fluid flow at small Reynolds numberse. We find the limiting calculated value of the turbulent Reynolds number at which wall terms must be taken into consideration.

In [1] a system of differential equations of a turbulent boundary layer was considered that consisted of equations of momentum, continuity, turbulence energy $E = \langle u_i u_i \rangle/2$, second moments of velocity pulsations $\langle u_1 u_2 \rangle$, and total rate of turbulence energy dissipation:

$$\varepsilon = \frac{\nu}{2} \sum_{ij} \left\langle \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle = \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle + \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right\rangle = \varepsilon_1 + \varepsilon_2 . \tag{1}$$

It was also noted that if one takes into account the total dissipation of turbulence energy, rather than its isotropic part ε_1, then in the differential equations for E and ε wall terms appear that, after some simplifications, take the form suggested in [2]. Moreover, in [1] the equation for $\langle u_1 u_2 \rangle$ was used in the form presented in [3], where there are no wall terms whatsoever.

The use of an equation for the total rate of turbulence energy dissipation rather than its isotropic part is essential in boundary layer calculations, since in some cases, as shown in [1], the anisotropic part of dissipation ε_2 in the wall layer is much greater than ε_1. As follows from the logic of [1], this is associated with the existence of significant wall terms in equations for turbulence energy and total turbulence energy dissipation.

The aim of the present work is determination of the wall term in the equation for $\langle u_1 u_2 \rangle$, determination of a convenient form of its statement for calculations and of the conditions under which the wall terms should be taken into account in practical calculations of a turbulent boundary layer.

Let us write the equation for the correlation of velocity pulsations $\langle u_i u_j \rangle$ in the form given in [4]:

$$\frac{d \langle u_i u_j \rangle}{dt} = \frac{\partial}{\partial x_k} \left[\nu \frac{\partial \langle u_i u_j \rangle}{\partial x_k} - \langle u_i u_j u_k \rangle \right] -$$

$$- \frac{1}{\rho} \left(\frac{\partial}{\partial x_i} \langle q u_j \rangle + \frac{\partial}{\partial x_j} \langle q u_i \rangle \right) + \frac{1}{\rho} \left\langle q \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\rangle -$$

$$- \langle u_j u_k \rangle \frac{\partial U_i}{\partial x_k} - \langle u_i u_k \rangle \frac{\partial U_j}{\partial x_k} - 2\nu \left\langle \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l} \right\rangle . \tag{2}$$

Here summation is performed over repeated indices. The terms that in Eq. (2) should be identified with the wall term have the form

$$\varepsilon_3 = - 2\nu \left\langle \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l} \right\rangle . \tag{3}$$

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For the plane case $i = 1$, $j = 2$, $l = 1, 2$; consequently,

$$\varepsilon_3 = -2\nu \left\langle \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_1} \right\rangle - 2\nu \left\langle \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_2} \right\rangle. \tag{4}$$

Using the continuity equation $\partial u_\alpha / \partial x_\alpha = 0$, we transform Eq. (4) to the form

$$\varepsilon_3 = 2\nu \left\langle \frac{\partial u_2}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right\rangle + 2\nu \left\langle \frac{\partial u_1}{\partial x_2} \frac{\partial u_1}{\partial x_1} \right\rangle = 2\nu \left\langle \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_j} \right\rangle. \tag{5}$$

Thus, proceeding from Eqs. (3) and (5), the wall term in Eq. (2) is

$$\varepsilon_3 = \nu \left\langle \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_j} - \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l} \right\rangle. \tag{6}$$

In [3] it is assumed that the approximation of ε_3 has the same form as the term $\langle q((\partial u_i/\partial x_j) + (\partial u_j/\partial x_i))\rangle$ in Eq. (2); therefore, their sum is replaced by the expression $-C_1(\varepsilon/E)\langle u_i u_j\rangle$. However, while this is valid for the above term with pressure pulsations, the quantity ε_3 requires a more thorough analysis.

Proceeding from Eq. (6), the value of ε_3 for plane flow can be transformed to

$$\varepsilon_3 = \nu \left\langle \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right) \right\rangle. \tag{7}$$

Squaring both sides of Eq. (7) and using the continuity equation, we obtain

$$\varepsilon_3^2 = \nu^2 \left\langle \left[\left(\frac{\partial u_1}{\partial x_2}\right)^2 - 2\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \left(\frac{\partial u_2}{\partial x_1}\right)^2\right] 4 \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \tag{8}$$

Here the expression in square brackets is equal to the difference between the instantaneous values of the isotropic and anisotropic parts of the turbulence energy dissipation rate:

$$\varepsilon_1' - \varepsilon_2' = \nu \left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right). \tag{9}$$

Moreover, proceeding from [5], the instantaneous value of the total dissipation of turbulence energy

$$\varepsilon' = \varepsilon_1' + \varepsilon_2' = 15\nu \left(\frac{\partial u_1}{\partial x_1}\right)^2. \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8), we obtain

$$\varepsilon_3^2 = \frac{4}{15}\langle(\varepsilon_1')^2 - (\varepsilon_2')^2\rangle = \frac{4}{15}\langle\varepsilon'(\varepsilon' - 2\varepsilon_2')\rangle. \tag{11}$$

Up to now, we have carried out only identity transformations of the quantity ε_3. Now, let us reduce Eq. (11) to a form convenient for calculations.

We replace the instantaneous values of energy dissipation by mean values; moreover, we use the expression for ε_2 obtained in [1]:

$$\varepsilon_2 = -\frac{2}{3}\nu\frac{\partial^2 E}{\partial x_2^2}. \tag{12}$$

After substitution of Eq. (12) into Eq. (11) and simple transformations, the wall term ε_3 in the equation for $\langle u_1 u_2\rangle$ takes the form

$$\varepsilon_3 = \frac{2}{\sqrt{15}} \left[\varepsilon \left(\varepsilon + \frac{4}{3} \nu \frac{\partial^2 E}{\partial x_2^2} \right) \right]^{1/2} . \tag{13}$$

Now, we write a total system of equations for calculating a steady-state turbulent boundary layer on a plane wall with allowance for Eq. (13):

momentum equation

$$\rho U_1 \frac{\partial U_1}{\partial x_1} + \rho U_2 \frac{\partial U_1}{\partial x_2} = - \frac{dp}{dx_1} + \frac{\partial}{\partial x_2} \left(\mu \frac{\partial U_1}{\partial x_2} - \rho \langle u_1 u_2 \rangle \right) ; \tag{14}$$

continuity equation

$$\frac{\partial \rho U_1}{\partial x_1} + \frac{\partial \rho U_2}{\partial x_2} = 0 ; \tag{15}$$

turbulence energy equation

$$\rho U_1 \frac{\partial E}{\partial x_1} + \rho U_2 \frac{\partial E}{\partial x_2} = \frac{\partial}{\partial x_2} \left[\left(\mu + \frac{\mu_t}{\sigma_E} \right) \frac{\partial E}{\partial x_2} \right] - \rho \langle u_1 u_2 \rangle \frac{\partial U_1}{\partial x_2} - \rho \varepsilon - \frac{2}{3} \mu \frac{\partial^2 E}{\partial x_2^2} ; \tag{16}$$

equation of the turbulence energy dissipation rate

$$\rho U_1 \frac{\partial \varepsilon}{\partial x_1} + \rho U_2 \frac{\partial \varepsilon}{\partial x_2} = \frac{\partial}{\partial x_2} \left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_2} \right] -$$

$$- \frac{\varepsilon}{E} \rho \langle u_1 u_2 \rangle \frac{\partial U_1}{\partial x_2} - 2\rho \frac{\varepsilon^2}{E} - 2\mu \frac{\partial \langle u_1 u_2 \rangle}{\partial x_2} \frac{\partial^2 U_1}{\partial x_2^2} ; \tag{17}$$

equation for the second moments of velocity pulsations

$$\rho U_1 \frac{\partial \langle u_1 u_2 \rangle}{\partial x_1} + \rho U_2 \frac{\partial \langle u_1 u_2 \rangle}{\partial x_2} = \frac{\partial}{\partial x_2} \left[\left(\mu + \frac{\mu_t}{\sigma_\tau} \right) \frac{\partial \langle u_1 u_2 \rangle}{\partial x_2} \right] -$$

$$- C_1 \left(\frac{\varepsilon}{E} \rho \langle u_1 u_2 \rangle + C_2 \rho E \frac{\partial U_1}{\partial x_2} \right) + \frac{2}{\sqrt{15}} \rho \left[\varepsilon \left(\varepsilon + \frac{4}{3} \nu \frac{\partial^2 E}{\partial x_2^2} \right) \right]^{1/2} . \tag{18}$$

The system of differential equations (14)-(18) makes it possible to calculate a turbulent boundary layer on a flat plate in a greater detail than in [1-3], since in this case all effects are included that are associated with a noticeable influence of the wall on fluid flow at low Reynolds numbers. For practical calculations it is necessary to find the conditions under which the wall terms in Eqs. (16), (17), and (18) are significant. To elucidate this, we will consider the physical meaning of the quantity ε_3^2 in Eq. (11). The form of notation of this quantity makes it possible to conclude that it characterizes the difference between the dissipations of the isotropic and anisotropic parts of turbulence.

In the case of the large turbulence Reynolds number $R = L_g \sqrt{E} / \nu$ the turbulence energy dissipates mainly due to the decomposition of small isotropic vortices according to Kolmogorov's theory of "local isotropy" [6], and the total dissipation of turbulence energy is expressed by the relation [6, p. 180]

$$\varepsilon_1 \approx \varepsilon = \frac{3}{2} A \frac{U'^3}{L_f} . \tag{19}$$

Taking into account that $U' = \sqrt{2E/3}$ and assuming on the average that $A = 1.1$, we obtain

$$\varepsilon_1 \approx 0.9 \frac{E^{3/2}}{L_f}. \tag{20}$$

Using the relationship $L_g = 0.5L_f$ [4, p. 203], we find the total dissipation of turbulence energy

$$\varepsilon_1 \approx 0.45 \frac{E^{3/2}}{L_g} = 0.45\nu R \frac{E}{L_g^2}. \tag{21}$$

At small Reynolds numbers the dissipation of anisotropic vortices is observed which is manifested mainly in the wall region of fluid flow. Since the wall terms in Eqs. (16) and (17) characterize the anisotropic part of dissipation, in the case of $\varepsilon_2 \ll \varepsilon_1$ they can be neglected in Eqs. (16)-(18). On the other hand, when turbulence degenerates in the wall region, the decisive role in the total dissipation of turbulence energy is played by its anisotropic part $\varepsilon_2 \gg \varepsilon_1$ [1]. Taking into account the fact that at the final stage of turbulence degeneration the dissipation of its energy is governed by the quantity $\varepsilon = -dE/dt$ and that $E = \mathrm{const}/t^{5/2}$, then, according to [6, p. 158], we obtain

$$\varepsilon = \frac{5}{2} \frac{E}{t}. \tag{22}$$

Using the expression for the coefficient of the longitudinal correlation of velocity pulsations between points located r apart $f(r, t) = \exp(-r^2/8\nu t)$ [4, p. 194], we find the longitudinal integral scale of turbulence [4, p. 66]:

$$L_f = \int_0^\infty f(r, t)\, dr = \frac{\sqrt{\pi}}{2} (8\nu t)^{1/2}. \tag{23}$$

Going over to the transverse scale of turbulence, we have

$$L_g^2 = \frac{\pi}{2} \nu t. \tag{24}$$

Combining relations (22) and (24), we find that at small values of R the anisotropic part of dissipation is equal to

$$\varepsilon_2 \approx \varepsilon = \nu \frac{5\pi}{4} \frac{E}{L_g^2}. \tag{25}$$

Comparing Eqs. (21) and (25), we determine the boundary turbulence Reynolds number R = 8.7.

Thus, if R < 8.7, then in calculations of the boundary layer on a plane wall it is necessary to take into account the wall terms in Eqs. (16)-(18).

As turbulence in a flow attains a developed stage and, consequently, the role of the wall terms in these equations is decreased, the numerical coefficient in the expression for ε, Eq. (25), changes from $5\pi/4$ to 0.45R, Eq. (21). As a result, the estimate obtained for the boundary turbulence Reynolds number R is minimal and can be used in calculations as a preliminary one.

NOTATION

U_i and u_i, components of mean velocity and velocity pulsations; p and q, mean pressure and pressure pulsations; ε and ε', mean and instantaneous values of turbulence energy dissipation rate; ρ, density; ν and μ, kinematic and dynamic viscosities; C_1, C_2, C_μ, σ_E, σ_ε, σ_τ, constant values; $\mu_t = C_\mu \rho E^2/\varepsilon$, turbulent dynamic viscosity; x_i, coordinate of flow; t, time; L_g, L_f, transverse and longitudinal integral scales of turbulence; U', mean-square rate of pulsations; A, coefficient equal to 0.8–1.4.

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